Tentamen Numerical Mathematics 2
November 6, 2006

Duration: 3 hours

N.B. Unless stated otherwise, the notation used is as in the book of Burden and Faires.
Use of pocket calculator is allowed.

Problem 1

Consider the system of ordinary differential equations (ODEs)

$$\frac{du_i}{dt} = (u_{i+1} - 2u_i + u_{i-1})/h^2$$

for $i = 1, \ldots, m-1$ with $h = 1/m$, $u_0(t) = u_m(t) = 0$, and $u_i(0) = \sin(\pi i h)$.

a) To which partial differential equation is this system consistent, and to what order of accuracy?

b. Locate the eigenvalues of the Jacobian matrix of the right-hand side using Gerschgorin's theorem.

c. Determine and sketch the regions of absolute stability for the Euler, Backward Euler and Trapezoidal method.

d. In the table below the result at $t = 0.5$ is given for the Euler method applied to the system of ODEs for $m = 10$ and the step size k is 0.0005 and 0.01, respectively. Explain the results using the region of absolute stability of the Euler method.

e. Do you expect the same difficulties for the backward Euler and Trapezoidal method? Explain why.

Problem 2

Consider the problem $Ax = b$, where A is a square strictly diagonally dominant matrix and b a known vector; x has to be solved.

a. Show that the matrix A is nonsingular.

x_i	$u(x_i, 0.5)$	$w_{i,1000}$ $k = 0.0005$	$\lvert u(x_i, 0.5) - w_{i,1000}\rvert$	$w_{i,50}$ $k = 0.01$	$\lvert u(x_i, 0.5) - w_i$
0.0	0	0		0	
0.1	0.00222241	0.00228652	6.411×10^{-5}	8.19876×10^{7}	8.199×10^{7}
0.2	0.00422728	0.00434922	1.219×10^{-4}	-1.55719×10^{8}	1.557×10^{8}
0.3	0.00581836	0.00598619	1.678×10^{-4}	2.13833×10^{8}	2.138×10^{8}
0.4	0.00683989	0.00703719	1.973×10^{-4}	-2.50642×10^{8}	2.506×10^{8}
0.5	0.00719188	0.00739934	2.075×10^{-4}	2.62685×10^{8}	2.627×10^{8}
0.6	0.00683989	0.00703719	1.973×10^{-4}	-2.49015×10^{8}	2.490×10^{8}
0.7	0.00581836	0.00598619	1.678×10^{-4}	2.11200×10^{8}	2.112×10^{8}
0.8	0.00422728	0.00434922	1.219×10^{-4}	-1.53086×10^{8}	1.531×10^{8}
0.9	0.00222241	0.00228652	6.511×10^{-5}	8.03604×10^{7}	8.036×10^{7}
1.0	0	0		0	

2.02.

b.) Describe the Jacobi method applied to this system and give the iteration matrix.

c.) Show that for any matrix A holds $\varrho(A) \leq ||A||_\infty$.

d.) Show that the Jacobi method converges for this problem.

e.) Show that the error in the $i-th$ iterate $x^{(i)}$ during the Jacobi iteration can be approximated by $||x - x^{(i)}|| \leq \frac{c}{1-c}||x^{(i)} - x^{(i-1)}||$ where $c = \frac{||x^{(i)}-x^{(i-1)}||}{||x^{(i-1)}-x^{(i-2)}||}$.

Problem 3

Consider the eigenvalue problem $Ax = \lambda x$, where A is a real symmetric matrix; for the eigenvalues holds $\lambda_1 > \lambda_2 \geq \lambda_3 \geq .... \geq \lambda_n \geq 0$.

~~a.~~ Describe the power method applied to this problem and show that it converges linearly with ratio $\frac{\lambda_2}{\lambda_1}$.

~~b.~~ If the Rayleigh-Ritz quotient, i.e. $(x, Ax)/(x, x)$, is used for the approximation of the eigenvalue the convergence proceeds with ratio $(\frac{\lambda_2}{\lambda_1})^2$; proof this. What does this mean compared to the standard power method?

c. A Householder matrix $I - 2ww^T$, with $||w||_2 = 1$ can be used to put

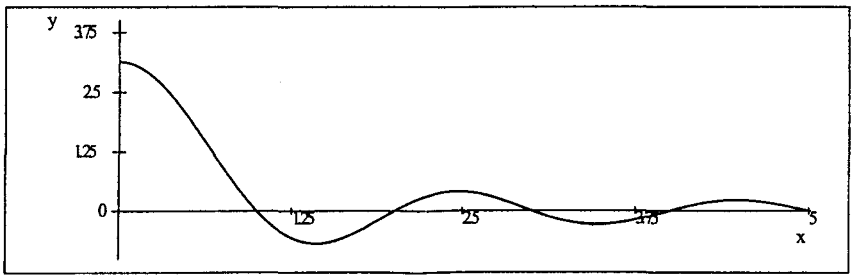
$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$$

into tridiagonal form using a similarity transformation. Give the w of the Householder matrix that will do the job. Explain the purpose of this procedure for the computation of eigenvalues.

d. Describe the QR method and explain why a shift is introduced to speed up the convergence.

Problem 4

Consider the function $f(x) = \sin(\pi x)/x$, which is plotted below on the interval [0,10].



We want to find the zero 3 of this function by a root finding method.

a.) What happens if we use the Newton method starting just right of the maximum near 2.5?

b.) Derive the ordinary differential equation (ODE) that has to be solved if a homotopy method based on $g(\lambda, x) = f(x) + (1 - \lambda)f(x_0)$ is applied to this problem and the interval on which it must be solved.

c.) Why will the homotopy method lead to the wanted zero, assuming we solve the ordinary differential equation exactly? What determines the accuracy of the found zero if we apply some ODE solver to the ODE derived in b?